Hardness of 4-colouring G-colourable graphs

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Joint work with:
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Graph homomorphisms

Definition (Graph Homomorphism)

Given $G = (V_G; E_G)$ and $H = (V_H; E_H)$ graphs, a **graph homomorphism** is a map $f : V_G \to V_H$ that respects edges, i.e. for all $(u, v) \in E_G$, $(f(u), f(v)) \in E_H$.

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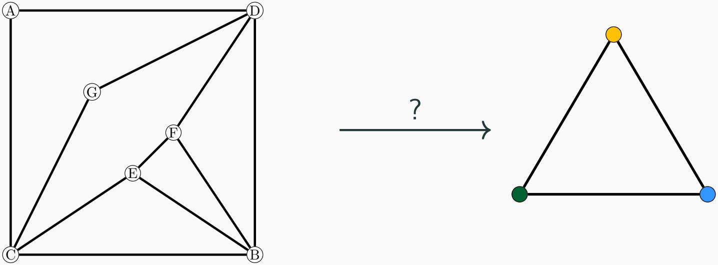

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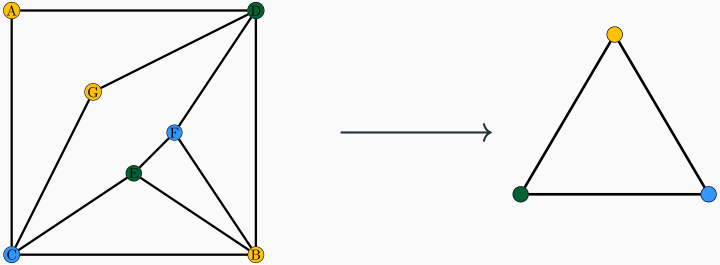

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Let G, H graphs such that $G \to H$. The (decision) Promise Graph Homomorphism Problem $\mathrm{PCSP}(G, H)$ is the following problem:

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Theorem (Krokhin, Opršal, Wrochna, Živný; '23)

For any 3-colourable non bipartite graph G, $\mathrm{PCSP}(G, K_3)$ *is NP-hard.*

Theorem (Avvakumov, Filakovský, Opršal, T., Wagner; '25+)

For any 4-colourable non bipartite graph G, $\mathrm{PCSP}(G, K_4)$ *is NP-hard.*

Proof Structure

- By a general algebraic theory of PCSPs[1], the complexity of PCSP(C_ℓ, K_4) is governed by its *polymorphisms*

$$f : C_\ell^n \to K_4.$$

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C_ℓ^n is categorical/tensor product: vertices of C_ℓ^n are n-tuples of vertices, edge between two tuples when each coordinate form an edge.

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- Via topology to $f\colon C_\ell^n \to K_4$ we associate a map $\phi(f) : \mathbb{Z}_2^n \to \mathbb{Z}_2$ of the form:

$$\phi(f)(x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_i x_i$$

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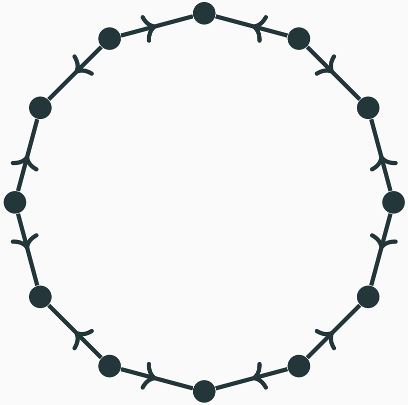
4. For $G = C_\ell$ and $G = K_4$, we can explicitly determine $\mathrm{Hom}(K_2, G)$.

When $\ell \geq 3$ is odd, $\Gamma_{4\ell} := \mathrm{Hom}(K_2, C_\ell)$ is topologically the circle S^1.



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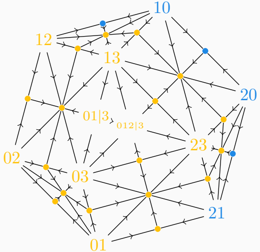

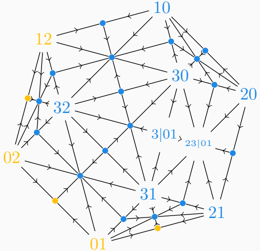

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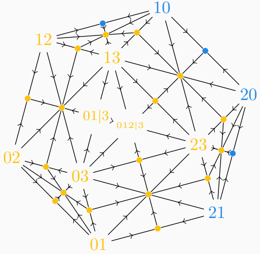

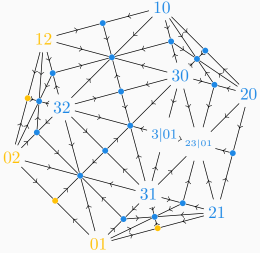

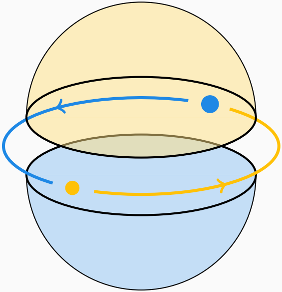

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We can classify $\mathbb{Z}_2$-maps $T^n \to Y$!

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Obs: Monomial maps correspond to linear maps $\mathbb{Z}_2^n \to \mathbb{Z}_2$.

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Prop 1: Different monomial maps are not equivalent.
Prop 2: Any $\mathbb{Z}_2$-map $f : T^n \to Y$ is equivalent to a monomial map.

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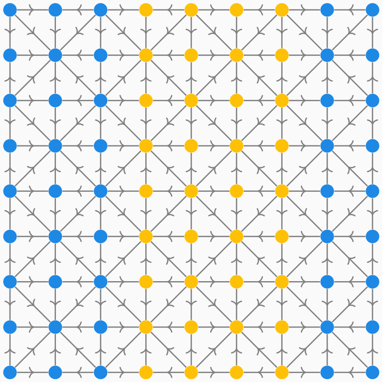
Building the minion homomorphism - Part II

How do we associate to $f : T^n \to Y$ the right monomial map?



Building the minion homomorphism - Part II

Use *degree* in direction i. First, for binary $g : T^2 \to Y$:

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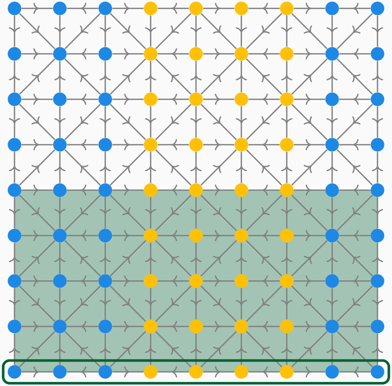

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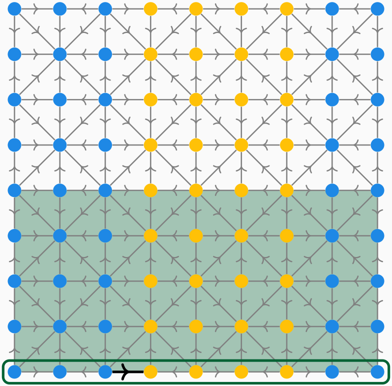
Ex: $\deg_1(g_1) = 1 + 0$



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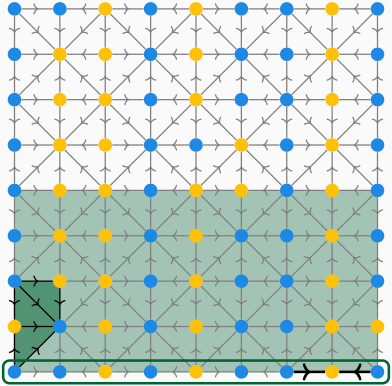

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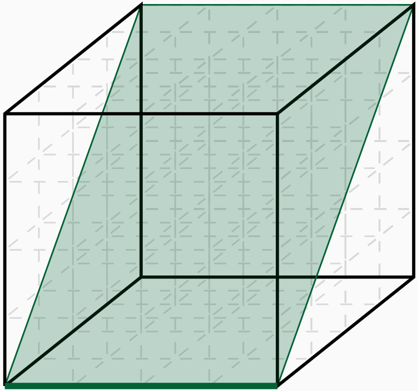

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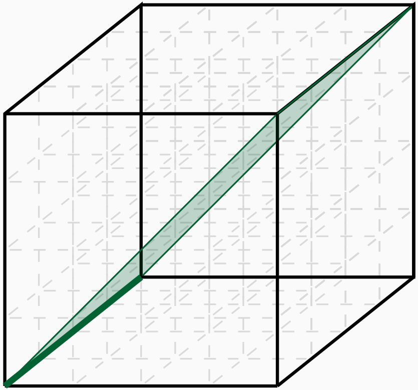

Building the minion homomorphism - Part II

Use *degree* in direction i. First, for binary $g : T^2 \to Y$:

$\deg_1(g) = \#\{[\bullet, \bullet] \text{ edges in } \square\} + \#\{[\bullet, \bullet, \bullet] \text{ triangles in } \blacksquare\} \pmod 2$

For $f : T^n \to Y$, set $g(x, y) = f(y, \ldots, y, x, y, \ldots, y)$, then:

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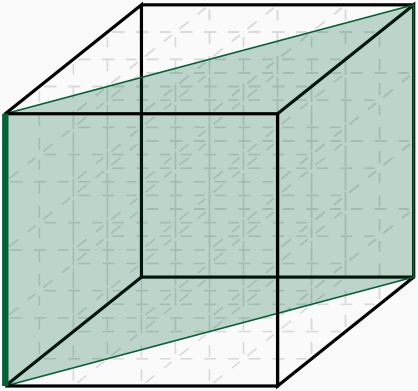
Ex: $\deg_3(f)$



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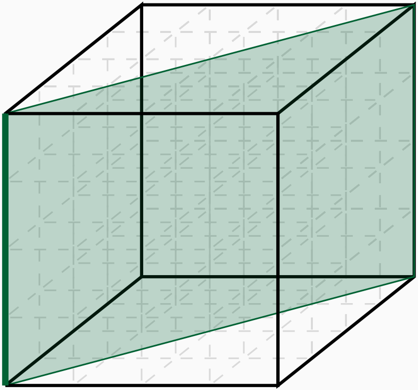
Finally, define $\phi : f \mapsto (\deg_1(f), \ldots, \deg_n(f))$.



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- More category theory (e.g. generalized nerve functors instead of Hom)?

Thank you!

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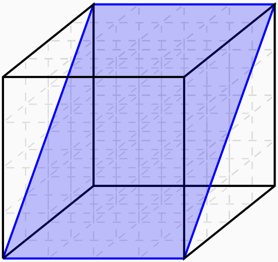

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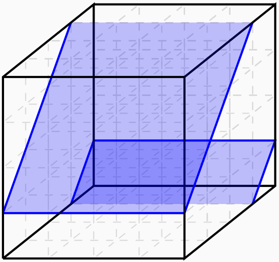

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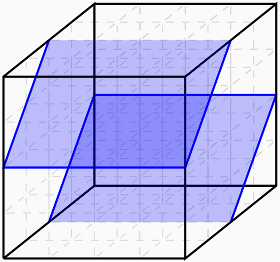

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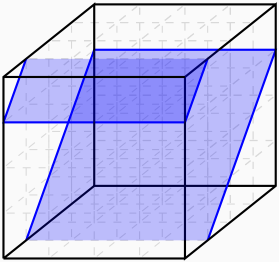

Bounding Essential Arity

Goal: Show that any simplicial map $f : \Gamma_L^n \to \Sigma^2$ can have at most $O(L^2)$ essential coordinates (i.e. $\deg_i(f) \neq 0$).

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Therefore,

$$n \leq 2CL^2$$

that is, any function in the image of ϕ has essential arity at most $O(L^2) \Rightarrow \mathrm{Im}\phi$ has bounded essential arity.