NP-hardness of linearly ordered 4-colouring of 3-colourable 3-uniform hypergraphs

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STACS 2024 - Track A

Prelude: Promise graph colourings

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Promise Graph Colouring problem (informal): Given a 3-colourable graph G, find a k-colouring ($k \geq 4$).

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Graph homomorphisms

Definition (Graph Homomorphism)

Given $G = (V_G; E_G)$ and $H = (V_H; E_H)$ graphs, a **graph homomorphism** is a map $f : V_G \to V_H$ that respects edges, i.e. for all $(u, v) \in E_G$, $(f(u), f(v)) \in E_H$.

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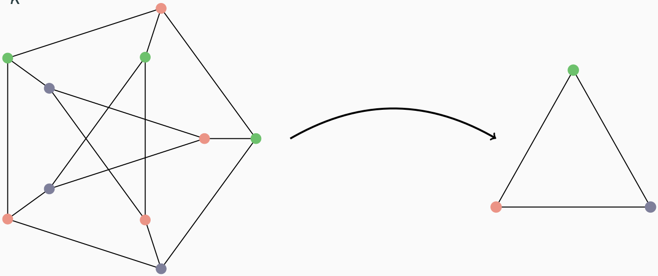
Example: k-colouring of G is the same as a graph homomorphism $G \to K_k$.



K_k = complete graph on k vertices.

Promise Graph Colourings

Let G, H graphs such that $G \to H$. The (decision) Promise Constraint Satisfaction Problem $\mathrm{PCSP}(G, H)$ is the following problem:

$$\begin{array}{rl} \textit{Input:} & \text{a graph } I \\ \textit{Output:} & \underline{\textbf{Yes}} \text{ if } I \to G \\ & \underline{\textbf{No}} \text{ if } I \nrightarrow H \end{array}$$

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Theorem (Krokhin, Opršal; '19 - Wrochna, Živný; '20)

For any 3-colourable non bipartite graph G, PCSP(G, K_3) *is NP-hard.*

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Example: if $\mathcal{H}$ is hypergraph, then a k-colouring is a homomorphism $\mathcal{H} \to \mathcal{K}_k = ([k]; \{E \subseteq [k] \mid |E| \geq 2\})$.

Hardness of Promise Hypergraph colourings

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Theorem (Dinur, Regev, Smyth; '05)

The problem $\mathrm{PCSP}(\mathcal{K}_k, \mathcal{K}_\ell)$ *for 3-uniform hypergraphs is NP-hard for any* $\ell \geq k \geq 2$.

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Giving an LO k-colouring for 3-uniform $\mathcal{H}$ is the same as homomorphism $\mathcal{H} \to \mathrm{LO}_k$.

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- $\mathrm{PCSP}(\mathrm{LO}_k, \mathrm{LO}_\ell)$ reduces to $\mathrm{PCSP}(\mathrm{LO}_{k+1}, \mathrm{LO}_{\ell+1})$ for any $\ell \geq k \geq 2$ [Nakajima, Živný; '22]

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- $\mathrm{PCSP}(K_k, K_\ell)$ reduces to $\mathrm{PCSP}(\mathrm{LO}_{k+1}, \mathrm{LO}_{\ell+1})$ for any $\ell \geq k \geq 3$ so it is NP-hard for
    - $k \geq 3$ and $\ell = 2k - 1$ [Bulín, Krokhin, Opršal; '19]
    - $k \geq 6$ and $\ell = \binom{k}{\lfloor k/2 \rfloor}$ [Wrochna, Živný; '20]

Our Result

Theorem (Filakovský, Nakajima, Opršal, T., Wagner)

The problem $\mathrm{PCSP}(\mathrm{LO}_3, \mathrm{LO}_4)$ *for 3-uniform hypergraphs is NP-hard.*

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- It is not covered by the previous cases;
- Proof uses topological methods, extending the approach used for $\mathrm{PCSP}(G, K_3)$.

Polymorphisms and minion homomorphisms

Polymorphisms

Given $\mathcal{A}, \mathcal{B}$ 3-uniform hypergraphs, their product $\mathcal{A} \times \mathcal{B}$ is the 3-uniform hypergraph with vertex set $A \times B$ and hyperedges

$$\mathcal{E}_{\mathcal{A}\times\mathcal{B}} = \left\{ ((a_1, b_1), (a_1, b_1), (a_1, b_1)) \in (A \times B)^3 \;\middle|\; \begin{array}{l} (a_1, a_2, a_3) \in \mathcal{E}_{\mathcal{A}} \\ (b_1, b_2, b_3) \in \mathcal{E}_{\mathcal{B}} \end{array} \right\}$$

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Definition (Polymorphism)

Let $(\mathcal{A}, \mathcal{B})$ a PCSP template, then a **polymorphism** of arity n is an homomorphism $h : \mathcal{A}^n \to \mathcal{B}$.

The set of all polymorphisms is $\mathrm{Pol}(\mathcal{A}, \mathcal{B})$.

Minion Homomorphisms

If we have a polymorphism, we can construct a new one by identifying coordinates and adding non-essential ones.

Example: if $f : \mathcal{A}^5 \to \mathcal{B}$ is a polymorphism, then $g : \mathcal{A}^3 \to \mathcal{B}$ defined as

$$g(x, y, z) = f(x, x, y, x, y)$$

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Def: Let $f : \mathcal{A}^n \to \mathcal{B}$ a polymorphism and $\pi : [n] \to [m]$ a map; the $\boldsymbol{\pi}$**-minor** of f is the polymorphism $\mathcal{A}^m \to \mathcal{B}$ defined as

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A **minion homomorphism** is a map $\eta : \mathrm{Pol}(\mathcal{A}, \mathcal{B}) \to \mathrm{Pol}(\mathcal{C}, \mathcal{D})$ that preserves arity and commutes with taking minors.

Minon Homomorphisms and reductions

Theorem (Bulín, Krokhin, Opršal, '19)

Let $(\mathcal{A}, \mathcal{B})$ *and* $(\mathcal{A}', \mathcal{B}')$ *be two PCSPs. If there is a minion homomorphism* $\mathrm{Pol}(\mathcal{A}', \mathcal{B}') \to \mathrm{Pol}(\mathcal{A}, \mathcal{B})$*, then there is a log-space reduction from* $\mathrm{PCSP}(\mathcal{A}, \mathcal{B})$ *to* $\mathrm{PCSP}(\mathcal{A}', \mathcal{B}')$*.*

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Rmk: Pol(3-SAT) is equivalent to the projections $\mathscr{P}_3 = \bigcup_n \{\pi_i : [3]^n \to [3] \mid \pi_i(x_1, \ldots, x_n) = x_i\}$.

Topology

From hypergraphs to topology: Hom complexes

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We fix a test 3-uniform hypergraph R_3 with a specific symmetry (cyclic group of order 3) and study $\mathrm{Hom}(R_3, -)$. Definition is rather technical, the key properties we will use are:

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2. for any $\mathcal{A} \to \mathcal{B}$, there is a corresponding continuous map $\mathrm{Hom}(R_3, \mathcal{A}) \to \mathrm{Hom}(R_3, \mathcal{B})$ respecting the symmetry
3. up to a standard notion of topological equivalence (homotopy),

$$\mathrm{Hom}(R_3, \mathcal{A}^n) \simeq (\mathrm{Hom}(R_3, \mathcal{A}))^n .$$

Building the minion homomorphism - Part I

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Understanding such continuous map up to topological equivalence is still complicated, we simplify by studying the composition $\eta(f)$

$$\begin{tikzcd} T^n = (S^1)^n \arrow[r] \arrow[rrr, bend right, "\eta(f)"'] & \mathrm{Hom}(R_3, \mathrm{LO}_3^n) \arrow[r, "f_*"] & \mathrm{Hom}(R_3, \mathrm{LO}_4) \arrow[r] & P^2 \end{tikzcd}$$

where P^2 is a suitable "nice" space (Eilenberg-MacLane space).

Building the minon homomorphism - Part II

Symmetry preserving maps from $T^n = (S^1)^n$ to P^2 up to topological equivalence can be classified:

$$[T^n, P^2] \simeq \{\phi : \mathbb{Z}_3^n \to \mathbb{Z}_3 \mid \phi(1, \ldots, 1) = 1\}$$

($\phi(1, 0, \ldots, 0) =$ "winding number" of f when "moving along" the first coordinate...)

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Symmetry preserving maps from $T^n = (S^1)^n$ to P^2 up to topological equivalence can be classified:

$$[T^n, P^2] \simeq \{\phi : \mathbb{Z}_3^n \to \mathbb{Z}_3 \mid \phi(1, \ldots, 1) = 1\}$$

($\phi(1, 0, \ldots, 0) =$ "winding number" of f when "moving along" the first coordinate...)

Lemma: The assignment

$$f \in \mathrm{Pol}^{(n)}(\mathrm{LO}_3, \mathrm{LO}_4) \mapsto \xi(f) \in [T^n, P^2] \simeq \mathscr{Z}_3^{(n)}$$

($\mathscr{Z}_3 =$ affine maps over $\mathbb{Z}_3$) respects minors, thus it is a minion homomorphism.

$[X, Y] =$ symmetry preserving maps $X \to Y$ up to topological equivalence.

Wrapping up

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Hence, $\xi(\mathrm{Pol}(\mathrm{LO}_3, \mathrm{LO}_4)) \subseteq \mathscr{P}_3$ as claimed.

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Vague question:	Which kind of PCSPs are suitable to be studied via topology?

Thank You!