8-Partitioning Points in 3D, and Efficiently Too

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ISTA

Joint work with:
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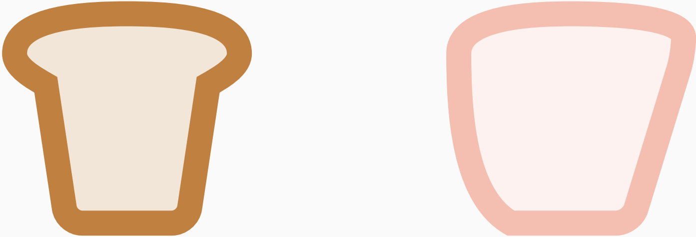
Prelude: Ham-Sandwich Theorem



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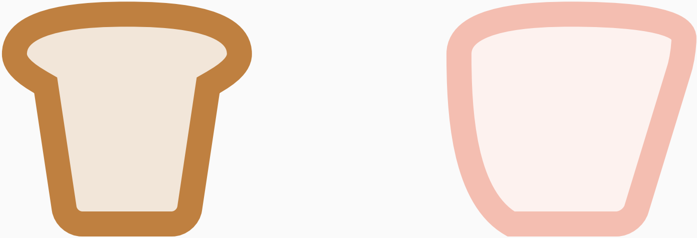

Theorem (Ham-Sandwich Theorem)

Let $\mu_1, \ldots, \mu_d$ *nice finite measures*[1] *on* $\mathbb{R}^d$. *Then there is an affine hyperplane* $H = \{p \in \mathbb{R}^d \mid \langle x, p \rangle = a\}$ *that simultaneously bisects all the measures; i.e., for any* $i \leq d$,

$$\mu_i(\{p \in \mathbb{R}^d \mid \langle x, p \rangle > a\}) = \mu_i(\{p \in \mathbb{R}^d \mid \langle x, p \rangle < a\})$$

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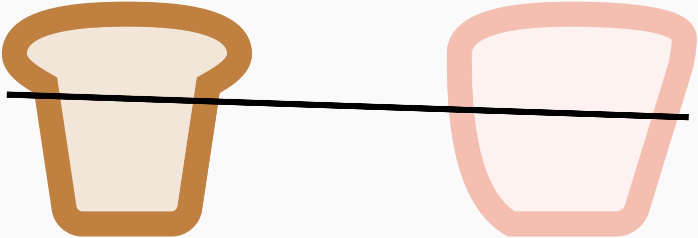

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Grünbaum's Partitioning Problem (Continuous Version)

Problem (Grünbaum; '60)

Given a nice probability measure on $\mathbb{R}^d$, is it possible to find a d-tuple of affine hyperplanes such that the total mass of every open orthant is $\frac{1}{2^d}$?

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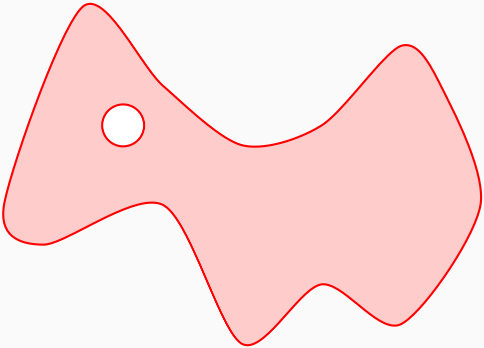

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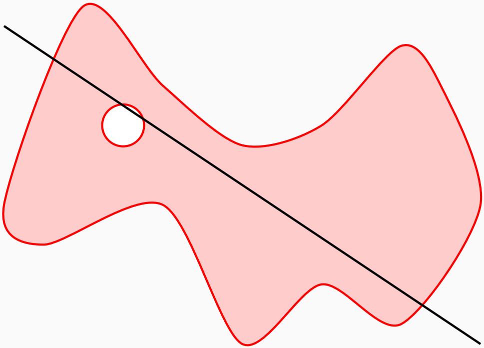

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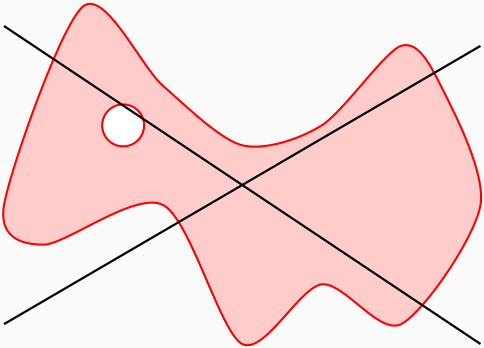

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It is always possible to find a triple of planes where the intersection line of two of them has a prescribed direction.

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$d = 4$: the problem is still open. It is not known if an equipartition always exists.

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$d \geq 5$: the problem is overconstrained (d^2 degrees of freedom, $2^d - 1$ constraints). Explicit counterexample due to [Avis; '86].

Grünbaum's Partitioning Problem (Discrete Version)

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Given a set $P \subseteq \mathbb{R}^d$ of n points in general position, is it always possible to find a d-tuple of affine hyperplanes such that every open orthant contains at most $\left\lfloor \frac{n}{2^d} \right\rfloor$ points in P?

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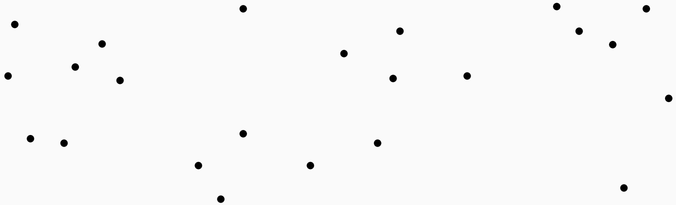

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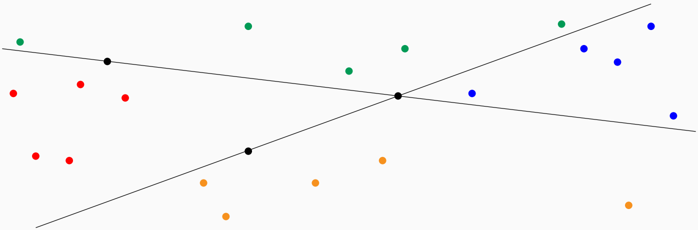

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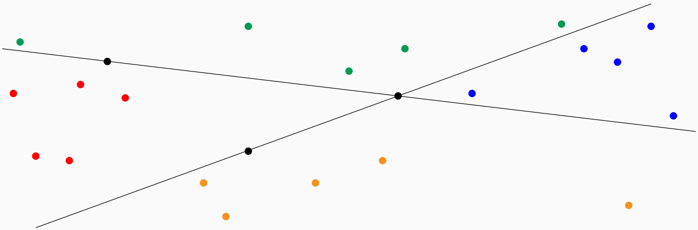

In general, existence of an equipartition for the continuous problem implies existence for discrete version.

Computing 8-partitions

Computational Complexity of the discrete Grünbaum problem

Problem

Let P be a set of n points in general position in $\mathbb{R}^3$, compute an 8-partition, i.e. a triple of planes (H_1, H_2, H_3) such that every open orthant contains at most $\left\lfloor \frac{n}{8} \right\rfloor$ points in P.

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In the '80, an algorithm[2] that computes in $O(n^6)$ a solution with prescribed normal for one of the planes was obtained.

[2] [Edelsbrunner; '86] and [Yao, Dobkin, Edelsbrunner, Paterson; '89]

Speed Up

Theorem (Aronov, Basit, Ramesh, T., Wagner; '24+)

Let $P \subseteq \mathbb{R}^3$ a set of n points in general position and $v \in S^2$. Then there is an algorithm that computes an eight-partition (H_1, H_2, H_3) of P with v the normal vector of H_1 in time $O^*(nh_2(n)) \leq O^*(n^{\frac{7}{3}})$; where $O^*(\cdot)$ hides polylog factors and $h_2(n) =$ max number of halving lines of a planar set of n points.

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Note: the asymptotic behaviour of $h_2(n)$ is not known. Best bounds are:

- $O(n^{\frac{4}{3}})$ [Dey; '97];
- $\Omega(ne^{\sqrt{\log n}})$ [Tóth; '01]

The Algorithm

Preliminaries

Properties of the Point Set

💡: We start by finding a plane that bisects P and it has v as its normal. This divides P in two sets R (points above) and B (points below); we can search for a solution among pairs of planes that simultaneously bisect both R and B.

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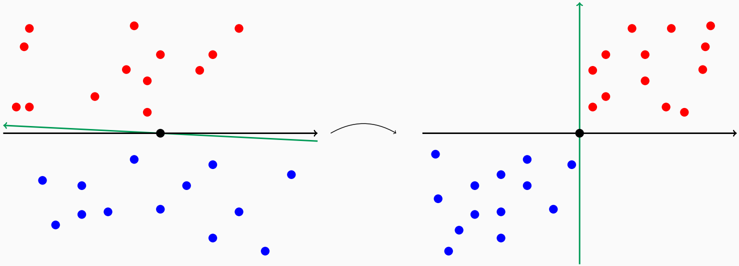

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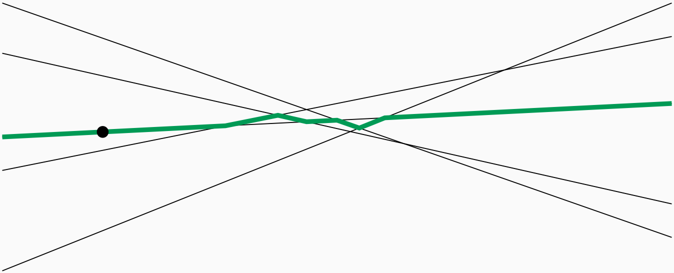
A (primal) plane bisects R (resp. B) iff the corresponding dual point has half of the planes in $\mathcal{A}(R)$ (resp. $\mathcal{A}(B)$) above and half below, i.e. it lies on the median level.



The Intersection Curve

Any plane in a solution has to simultaneously bisects both R and B, hence its dual point has to belong to L, the intersection of the median levels of $\mathcal{A}(R)$ and $\mathcal{A}(B)$.

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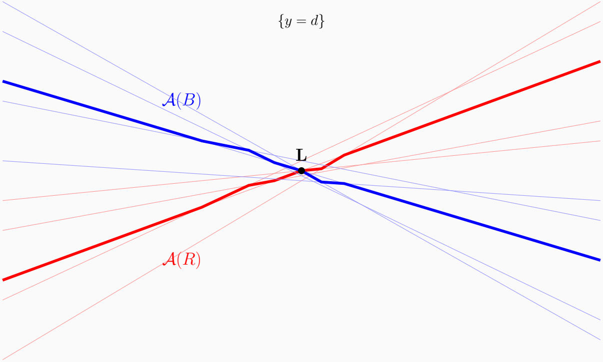
Proof [💡]:



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- If R and B are just in general position, m is $\Theta(h_3(n))$, where $h_3(n)$ is the maximum number of halving planes in a set of n points in $\mathbb{R}^3$. Best known bound is $O(n^{\frac{5}{2}})$ [Sharir, Smorodinsky, Tardos; '01].

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- Under our separation assumptions on R and B, m is $\Theta(n \underbrace{h_2(n)}_{O(n^{4/3})}) = O(n^{7/3})$.

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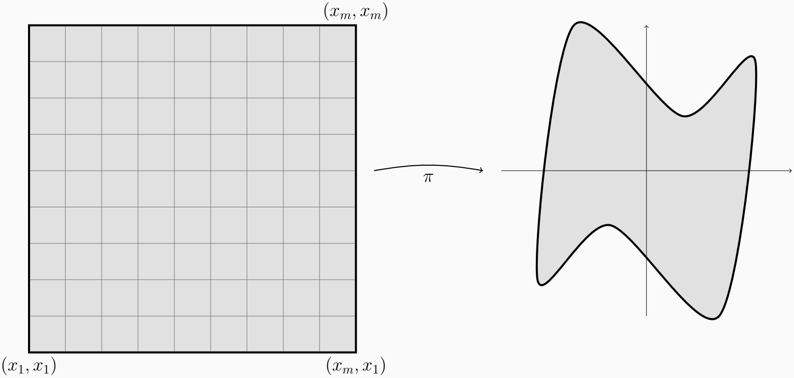
Lemma

Two planes h_1, h_2 simultaneously 4-partition R and B if and only if their duals $h_1^\star, h_2^\star$ lie on L and $X(h_1^\star, h_2^\star) = Y(h_1^\star, h_2^\star) = 0$.

The Geometric Idea

Using the alternating sums, we can define a map $\pi : L^2 \to \mathbb{R}^2$, $(p, q) \mapsto (X(p, q), Y(p, q))$.



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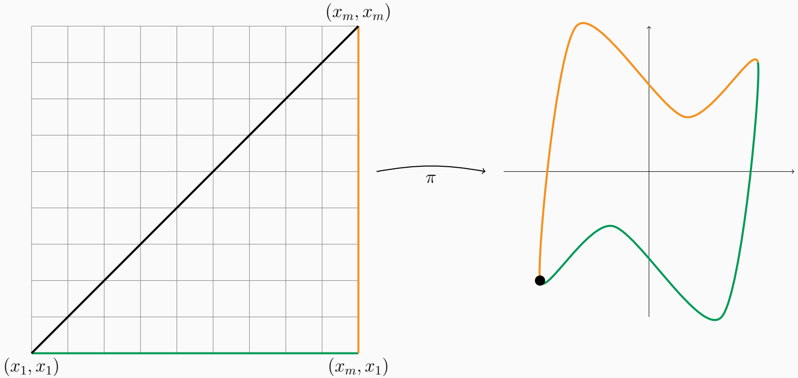

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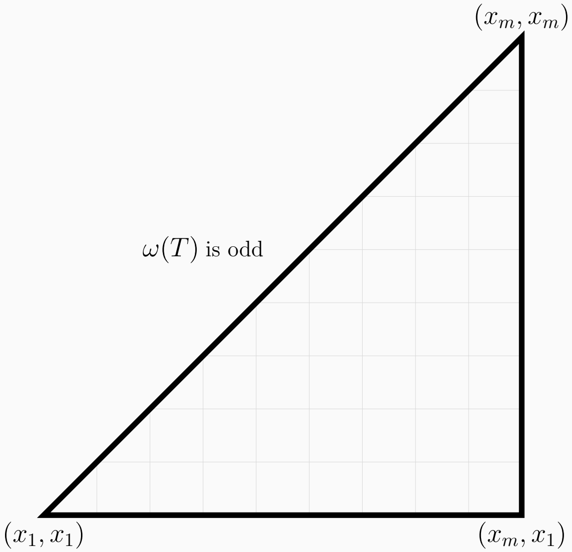

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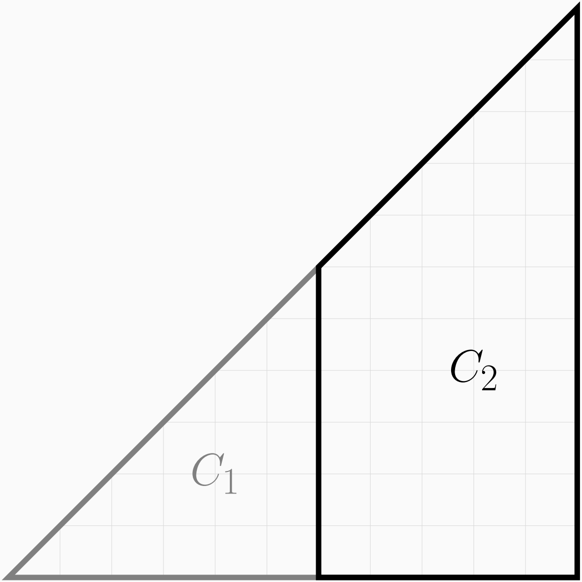

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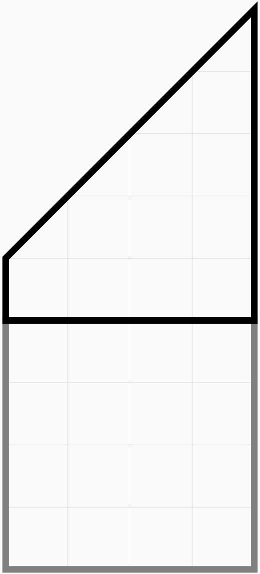

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Since m is $\Theta(nh_2(n))$ and $h_2(n)$ is $O(n^{\frac{4}{3}})$ we have the desired $O^*(n^{\frac{7}{3}})$ running time.

Where to go from here...

Hard questions:

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(Potentially) easier questions:

- Is it possible to compute a solution in $o(nh_2(n))$?
- Find an algorithm for the other "types" of equipartitions (e.g. orthogonality condition or prescribed intersection).

Thank You!